Sheet #1

1 Find the vector $\bar{A}$ directed from (2,-4,1) to (0,-2,0) in Cartesian coordinates and find the unit vector along $\bar{A}$

Answer

$$(2,-4,1) \xrightarrow{\bar{A}} (0,-2,0)$$

$$\bar{A} = (0-2)\bar{a}_x + \big(-2-(-4)\big)\bar{a}_y + (0-1)\bar{a}_z$$

$$\boldsymbol{\bar{A} = -2\bar{a}_x + 2\bar{a}_y - \bar{a}_z}$$

$$\bar{a}_A = \frac{\bar{A}}{|\bar{A}|}$$

$$|\bar{A}| = \sqrt{(-2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\bar{a}_A = \frac{-2\bar{a}_x + 2\bar{a}_y - \bar{a}_z}{3}$$

$$\boldsymbol{\bar{a}_A = \frac{-2}{3}\bar{a}_x + \frac{2}{3}\bar{a}_y - \frac{1}{3}\bar{a}_z}$$

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2 Show that $\bar{A} = 4\bar{a}_x - 2\bar{a}_y - \bar{a}_z$ and $\bar{B} = \bar{a}_x + 4\bar{a}_y - 4\bar{a}_z$ are perpendicular

Answer

The vectors $\bar{A}$ and $\bar{B}$ are perpendicular when

$$\bar{A}\cdot\bar{B} = 0$$

$$\bar{A}\cdot\bar{B} = |\bar{A}||\bar{B}|\cos\theta_{AB} = 0$$

$$\cos\theta_{AB} = 0 \Rightarrow \theta_{AB} = 90^\circ$$

$$\bar{A}\cdot\bar{B} = \big(4\bar{a}_x - 2\bar{a}_y - \bar{a}_z\big)\cdot\big(\bar{a}_x + 4\bar{a}_y - 4\bar{a}_z\big)$$

$$\bar{A}\cdot\bar{B} = (4\times 1) + (-2\times 4) + (-1\times -4)$$

$$\bar{A}\cdot\bar{B} = 4 - 8 + 4 = 0$$

$$\because \boldsymbol{\bar{A}\cdot\bar{B} = 0}$$

$$\therefore \boldsymbol{\bar{A} \perp \bar{B}}$$

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3 Determine the smaller angle between

$$\bar{A} = 2\bar{a}_x + 4\bar{a}_y \quad \text{and} \quad \bar{B} = 6\bar{a}_y - 4\bar{a}_z$$

using the cross product and also the dot product

Answer

Using Dot Product	**Using Cross Product**
$\bar{A}\cdot\bar{B} = \lvert\bar{A}\rvert\lvert\bar{B}\rvert\cos\theta_{AB}$ $\cos\theta_{AB} = \dfrac{\bar{A}\cdot\bar{B}}{\lvert\bar{A}\rvert\lvert\bar{B}\rvert}$ $\bar{A}\cdot\bar{B} = (2\bar{a}_x + 4\bar{a}_y)\cdot(6\bar{a}_y - 4\bar{a}_z)$ $\bar{A}\cdot\bar{B} = 4\ x\ 6 = 24$ $\lvert\bar{A}\rvert = \sqrt{(2)^2 + (4)^2} = \sqrt{4+16}$ $\lvert\bar{A}\rvert = \sqrt{20}$ $\lvert\bar{B}\rvert = \sqrt{(6)^2 + (-4)^2} = \sqrt{36+16}$ $\lvert\bar{B}\rvert = \sqrt{52}$ $\cos\theta_{AB} = \dfrac{24}{\sqrt{20}\sqrt{52}} = 0.744208$ $\theta_{AB} = \cos^{-1} 0.744208$ $\boldsymbol{\theta_{AB} = 41.9088°}$	$\bar{A}\times\bar{B} = \lvert\bar{A}\rvert\lvert\bar{B}\rvert\sin\theta_{AB}\,\bar{a}_n$ $\lvert\bar{A}\times\bar{B}\rvert = \lvert\bar{A}\rvert\lvert\bar{B}\rvert\sin\theta_{AB}$ $\sin\theta_{AB} = \dfrac{\lvert\bar{A}\times\bar{B}\rvert}{\lvert\bar{A}\rvert\lvert\bar{B}\rvert}$ $\bar{A}\times\bar{B} = (2\bar{a}_x + 4\bar{a}_y)\times(6\bar{a}_y - 4\bar{a}_z)$ $\bar{A}\times\bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 2 & 4 & 0 \\ 0 & 6 & -4 \end{vmatrix}$ $\bar{A}\times\bar{B} = \bar{a}_x[-16] - \bar{a}_y[-8] + \bar{a}_z[12]$ $\bar{A}\times\bar{B} = -16\bar{a}_x + 8\bar{a}_y + 12\bar{a}_z$ $\lvert\bar{A}\times\bar{B}\rvert = \sqrt{(-16)^2 + (8)^2 + (12)^2}$ $\lvert\bar{A}\times\bar{B}\rvert = \sqrt{256+64+144}$ $\lvert\bar{A}\times\bar{B}\rvert = \sqrt{464}$ $\lvert\bar{A}\rvert = \sqrt{20}$ $\lvert\bar{B}\rvert = \sqrt{52}$ $\sin\theta_{AB} = \dfrac{\sqrt{464}}{\sqrt{20}\sqrt{52}} = 0.667947$ $\theta_{AB} = \sin^{-1} 0.667947$ $\boldsymbol{\theta_{AB} = 41.9088°}$ or $\mathbf{138.0911°}$

$\boxed{4}$ Given $\bar{F} = (y-1)\bar{a}_x + 2x\bar{a}_y$, find the vector at (2,2,1) and its projection on $\bar{B} = 5\bar{a}_y - \bar{a}_y + 2\bar{a}_z$

Answer

At point (2,2,1) , $\bar{F} = (2-1)\bar{a}_x + 2\text{x}2\ \bar{a}_y$

$$\boldsymbol{\bar{F} = \bar{a}_x + 4\bar{a}_y}$$

$\text{Projection of } \bar{F} \text{ onto } \bar{B} = |\bar{F}|\cos\theta$

$\bar{F}\cdot\bar{B} = |\bar{F}||\bar{B}|\cos\theta$

$\text{Projection of } \bar{F} \text{ onto } \bar{B} = \dfrac{\bar{F}\cdot\bar{B}}{|\bar{B}|} = \bar{F}\cdot\bar{a}_B$

$$\text{Projection of } \bar{F} \text{ onto } \bar{B} = \frac{\bar{F}\cdot\bar{B}}{|\bar{B}|}$$

$\bar{F}\cdot\bar{B} = \left(\bar{a}_x + 4\bar{a}_y\right)\cdot\left(5\bar{a}_y - \bar{a}_y + 2\bar{a}_z\right)$ $\bar{F}\cdot\bar{B} = (1\text{x}5) + (4\text{x}-1) + (0\text{x}2) = 1$	$\lvert\bar{B}\rvert = \sqrt{(5)^2 + (-1)^2 + (2)^2} = \sqrt{30}$

$$\textbf{Projection of } \boldsymbol{\bar{F}} \textbf{ onto } \boldsymbol{\bar{B}} = \frac{\mathbf{1}}{\sqrt{\mathbf{30}}}$$

$\boxed{5}$ If $\bar{A} = \bar{a}_x + 2\bar{a}_y - 3\bar{a}_z$ and $\bar{B} = 2\bar{a}_x - \bar{a}_y + \bar{a}_z$

Determine :

a) The magnitude of projection of $\bar{B}$ on $\bar{A}$

b) The smallest angle between $\bar{A}$ and $\bar{B}$

c) The vector projection $\bar{A}$ onto $\bar{B}$

d) A unit vector perpendicular to the plane containing $\bar{A}$ and $\bar{B}$

Answer

$$\bar{A} \cdot \bar{B} = \left(\bar{a}_x + 2\bar{a}_y - 3\bar{a}_z\right) \cdot \left(2\bar{a}_x - \bar{a}_y + \bar{a}_z\right) = (1\text{x}2) + (2\text{x} - 1) + (-3\text{x}1)$$

$$\bar{\boldsymbol{A}} \cdot \bar{\boldsymbol{B}} = \mathbf{-3}$$

$$|\bar{A}| = \sqrt{(1)^2 + (2)^2 + (-3)^2} = \sqrt{1 + 4 + 9}$$

$$|\bar{\boldsymbol{A}}| = \sqrt{\mathbf{14}}$$

$$|\bar{B}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{4 + 1 + 1}$$

$$|\bar{\boldsymbol{B}}| = \sqrt{\mathbf{6}}$$

a) The magnitude of projection of $\bar{B}$ on $\bar{A}$

$$\text{Projection of } \bar{B} \text{ on } \bar{A} = \frac{\bar{B} \cdot \bar{A}}{|\bar{A}|} = \frac{-3}{\sqrt{14}}$$

$$\textbf{Magnitude Projection of } \bar{\boldsymbol{B}} \textbf{ onto } \bar{\boldsymbol{A}} = \frac{\mathbf{3}}{\sqrt{\mathbf{14}}}$$

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b) The smallest angle between $\bar{A}$ and $\bar{B}$

$$\theta_{AB} = \cos^{-1}\frac{\bar{A} \cdot \bar{B}}{|\bar{A}||\bar{B}|} = \cos^{-1}\frac{-3}{\sqrt{14}\sqrt{6}}$$

$$\boldsymbol{\theta_{AB}} = \mathbf{109.1066^\circ}$$

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c) The vector projection $\bar{A}$ onto $\bar{B}$

$$\text{Projection of } \bar{A} \text{ on } \bar{B} = \frac{\bar{A} \cdot \bar{B}}{|\bar{B}|} = \frac{-3}{\sqrt{6}}$$

$$\text{Vector Projection of } \bar{A} \text{ onto } \bar{B} = \frac{\bar{A}\cdot\bar{B}}{|\bar{B}|}\bar{a}_B$$

$$\text{Vector Projection of } \bar{B} \text{ onto } \bar{A} = \left(\frac{-3}{\sqrt{6}}\right)\bar{a}_B$$

$$\text{Vector Projection of } \bar{B} \text{ onto } \bar{A} = \left(\frac{-3}{\sqrt{6}}\right)\left(\frac{2\bar{a}_x - \bar{a}_y + \bar{a}_z}{\sqrt{6}}\right)$$

$$\textbf{Vector Projection of } \bar{\boldsymbol{B}} \textbf{ onto } \bar{\boldsymbol{A}} = -\bar{\boldsymbol{a}}_{\boldsymbol{x}} + \mathbf{0.5}\bar{\boldsymbol{a}}_{\boldsymbol{y}} - \mathbf{0.5}\bar{\boldsymbol{a}}_{\boldsymbol{z}}$$

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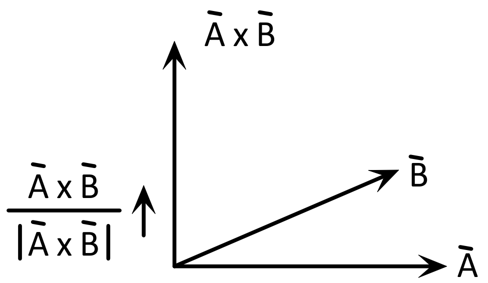
d) A unit vector perpendicular to the plane containing $\bar{A}$ and $\bar{B}$

There are two possible methods

$\bar{A}\times\bar{B} = \lvert\bar{A}\times\bar{B}\rvert\bar{a}_n$ $\bar{a}_n = \dfrac{\bar{A}\times\bar{B}}{\lvert\bar{A}\times\bar{B}\rvert}$	$\bar{A}\times\bar{B} = \lvert\bar{A}\rvert\lvert\bar{B}\rvert\sin\theta_{AB}\,\bar{a}_n$ $\bar{a}_n = \dfrac{\bar{A}\times\bar{B}}{\lvert\bar{A}\rvert\lvert\bar{B}\rvert\sin\theta_{AB}}$

$\bar{A}\times\bar{B} = \begin{vmatrix}\bar{a}_x & \bar{a}_y & \bar{a}_z\\ 1 & 2 & -3\\ 2 & -1 & 1\end{vmatrix}$ $\bar{A}\times\bar{B} = \bar{a}_x[-1] - \bar{a}_y[7] + \bar{a}_z[-5]$ $\bar{A}\times\bar{B} = -\bar{a}_x - 7\bar{a}_y - 5\bar{a}_z$	$\lvert\bar{A}\times\bar{B}\rvert = \sqrt{(-1)^2 + (-7)^2 + (-5)^2}$ $\lvert\bar{A}\times\bar{B}\rvert = \sqrt{1 + 49 + 25}$ $\lvert\bar{A}\times\bar{B}\rvert = \sqrt{75}$

$$\bar{a}_n = \frac{\bar{A}\times\bar{B}}{|\bar{A}\times\bar{B}|} = \frac{-\bar{a}_x - 7\bar{a}_y - 5\bar{a}_z}{\sqrt{75}}$$



Generally

$$\bar{\boldsymbol{a}}_{\boldsymbol{n}} = \pm\frac{\bar{\boldsymbol{a}}_{\boldsymbol{x}} + \mathbf{7}\bar{\boldsymbol{a}}_{\boldsymbol{y}} + \mathbf{5}\bar{\boldsymbol{a}}_{\boldsymbol{z}}}{\sqrt{\mathbf{75}}}$$

$\boxed{6}$ Given $\quad \bar{A} = \bar{a}_x + \bar{a}_y \quad , \quad \bar{B} = \bar{a}_x + 2\bar{a}_z \quad , \quad \bar{C} = 2\bar{a}_y + \bar{a}_z$

Find $(\bar{A} \times \bar{B}) \times \bar{C}$ and compare it with $\bar{A} \times (\bar{B} \times \bar{C})$, comment on the result.

Answer

$(\bar{A} \times \bar{B}) \times \bar{C}$	$\bar{A} \times (\bar{B} \times \bar{C})$
$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix}$ $\bar{A} \times \bar{B} = \bar{a}_x[2] - \bar{a}_y[2] + \bar{a}_z[-1]$ $\bar{A} \times \bar{B} = 2\bar{a}_x - 2\bar{a}_y - \bar{a}_z$ $(\bar{A} \times \bar{B}) \times \bar{C} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 2 & -2 & -1 \\ 0 & 2 & 1 \end{vmatrix}$ $(\bar{A} \times \bar{B}) \times \bar{C} = \bar{a}_x[0] - \bar{a}_y[2] + \bar{a}_z[4]$ $\boxed{(\bar{\boldsymbol{A}} \times \bar{\boldsymbol{B}}) \times \bar{\boldsymbol{C}} = -\boldsymbol{2}\bar{\boldsymbol{a}}_y + \boldsymbol{4}\bar{\boldsymbol{a}}_z}$	$\bar{B} \times \bar{C} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix}$ $\bar{B} \times \bar{C} = \bar{a}_x[-4] - \bar{a}_y[1] + \bar{a}_z[2]$ $\bar{B} \times \bar{C} = -4\bar{a}_x - \bar{a}_y + 2\bar{a}_z$ $\bar{A} \times (\bar{B} \times \bar{C}) = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 1 & 1 & 0 \\ -4 & -1 & 2 \end{vmatrix}$ $\bar{A} \times (\bar{B} \times \bar{C}) = \bar{a}_x[2] - \bar{a}_y[2] + \bar{a}_z[3]$ $\boxed{\bar{\boldsymbol{A}} \times (\bar{\boldsymbol{B}} \times \bar{\boldsymbol{C}}) = \boldsymbol{2}\bar{\boldsymbol{a}}_x - \boldsymbol{2}\bar{\boldsymbol{a}}_y + \boldsymbol{3}\bar{\boldsymbol{a}}_z}$

It is clear that

$$(\bar{A} \times \bar{B}) \times \bar{C} \neq \bar{A} \times (\bar{B} \times \bar{C})$$

So, when we are calculating the cross product of three vectors $\boxed{\bar{A} \times \bar{B} \times \bar{C}}$, brackets are needed to determine how to start calculating

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$\boxed{7}$ Find $\bar{A} \cdot \bar{B} \times \bar{C}$ for $\bar{A}$, $\bar{B}$, $\bar{C}$ of problem $\boxed{6}$ and compare it with $\bar{A} \times \bar{B} \cdot \bar{C}$
comment on the result

Answer

$\bar{A} \cdot \bar{B} \times \bar{C}$	$\bar{A} \times \bar{B} \cdot \bar{C}$
$\bar{B} \times \bar{C} = -4\bar{a}_x - \bar{a}_y + 2\bar{a}_z$ $\bar{A} \cdot \bar{B} \times \bar{C} = (1\text{x} - 4) + (1\text{x} - 1) + (0\text{x}2)$ $\bar{\boldsymbol{A}} \cdot \bar{\boldsymbol{B}} \times \bar{\boldsymbol{C}} = -\boldsymbol{5}$	$\bar{A} \times \bar{B} = 2\bar{a}_x - 2\bar{a}_y - \bar{a}_z$ $\bar{A} \times \bar{B} \cdot \bar{C} = (2\text{x}0) + (-2\text{x}2) + (-1\text{x}1)$ $\bar{\boldsymbol{A}} \times \bar{\boldsymbol{B}} \cdot \bar{\boldsymbol{C}} = -\boldsymbol{5}$

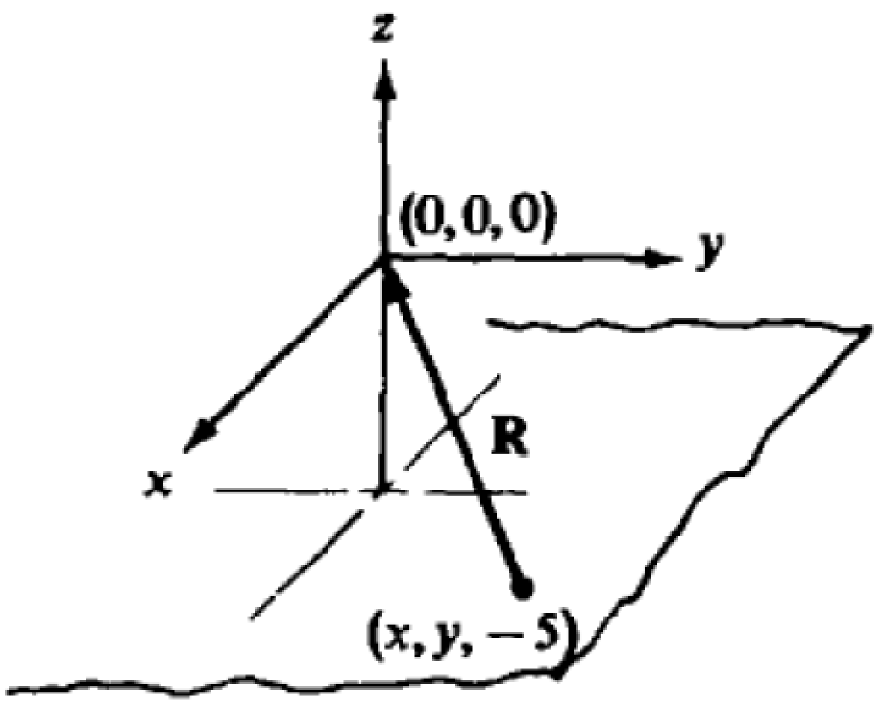
It is clear that

$$\bar{A} \cdot \bar{B} \times \bar{C} = \bar{A} \times \bar{B} \cdot \bar{C}$$

This is because both values are the volume of parallelepiped whose sides are $\bar{A}$, $\bar{B}$ and $\bar{C}$

$\boxed{8}$ Express the unit vector which is directed toward the origin from an arbitrary point on the plane $z = -5$

Answer



An arbitrary point on the plane $z = -5$, will have coordinates $(x, y, -5)$

$$\bar{R} = (0 - x)\bar{a}_x + (0 - y)\bar{a}_y + \big(0 - (-5)\big)\bar{a}_z$$

$$\bar{R} = -x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z$$

$$|\bar{R}| = \sqrt{(-x)^2 + (-y)^2 + (5)^2}$$

$$|\bar{R}| = \sqrt{x^2 + y^2 + 25}$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|}$$

$$\boldsymbol{\bar{a}_R = \frac{-x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z}{\sqrt{x^2 + y^2 + 25}}}$$

$\boxed{9}$ Given the two vectors $\bar{A} = -\bar{a}_x - 3\bar{a}_y - 4\bar{a}_z$, $\bar{B} = 2\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z$ and a point $C(1,3,4)$, Find

(a) $\bar{R}_{AB}$ (b) $|\bar{A}|$ (c) $\bar{a}_A$ (d) $\bar{a}_{AB}$

(e) a unit vector directed from C toward A

Answer

REPORT

$\boxed{10}$ A triangle is defined by three points $A\ (2,-5,1) B\ (-3,2,4) C\ (0,3,1)$ Find

a) $\bar{R}_{BC} \times \bar{R}_{BA}$
b) The area of the triangle
c) A unit vector perpendicular to the plane of the triangle

Answer

REPORT